A Swimmer's Round Trip Average Speed in a River

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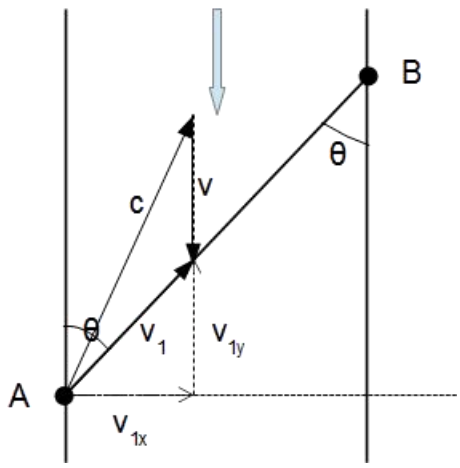
Settings:

A swimmer is trying to swim across a river, in a straight line from a point A on the left bank to a point B on the right bank, and then will return to A following the same path. The swimmer's speed in still water is c, and the water in the river is flowing at a speed of v.

Question:

What is his average speed in this round trip?



Solution:

To compensate for the influence of the water, the swimmer's swimming direction has to be different from his intended direction AB.

If we can get his upstream speed v_1 and downstream speed v_2, then the average speed can be easily obtained by

$$\frac{1}{v_1}+\frac{1}{v_2}=\frac{2}{v_{avg}}, \quad \text{or}$$

$$v_{avg}=\frac{2v_1v_2}{v_1+v_2} \qquad \text{(I)}$$

So our first step is to get v_1 and v_2.

Using pythagoras theorem, we have

$$c^2 = v_{1x}^2 + (v_{1y} + v)^2,$$

where $v_{1x} = v_1 \sin\theta$ and $v_{1y} = v_1 \cos\theta$.

Thus we get

$$c^2 = (v_1 \sin\theta)^2 + (v_1 \cos\theta + v)^2 \qquad (1.1)$$

$$c^2 = v_1^2 \sin^2\theta + v_1^2 \cos^2\theta + 2v_1 v \cos\theta + v^2$$

$$c^2 = v_1^2(\sin^2\theta + \cos^2\theta) + 2v_1 v \cos\theta + v^2$$

$$v_1^2 + 2v\cos\theta \cdot v_1 + v^2 - c^2 = 0$$

$$v_1 = \frac{-2v\cos\theta \pm \sqrt{(2v\cos\theta)^2 - 4(v^2 - c^2)}}{2}$$

$$v_1 = -v\cos\theta \pm \sqrt{(v\cos\theta)^2 + (c^2 - v^2)}$$

$$v_1 = -v\cos\theta \pm \sqrt{(v^2\cos^2\theta - v^2) + c^2}$$

$$v_1 = -v\cos\theta \pm \sqrt{c^2 - v^2\sin^2\theta}$$

When $v = 0$, $v_1 = \pm c$.

We know when the water is still, the swimmer's speed is c, so the + sign should be taken, thus

$$v_1 = -v\cos\theta + \sqrt{c^2 - v^2\sin^2\theta} \qquad (1.2)$$

For the return speed v_2, we have

$$c^2 = (v_2 \sin\theta)^2 + (v_2 \cos\theta - v)^2 \qquad (2.1)$$

Comparing it with (1.1) , we can see the only difference is the sign in front of v.

Replacing the v in (1.2) with $-v$ will give us v_2:

$$v_2 = v\cos\theta + \sqrt{c^2 - v^2\sin^2\theta} \qquad (2.2)$$

Put (1.2) and (2.2) into the average speed formula (I) :

$$v_{avg} = 2\frac{\left(-v\cos\theta + \sqrt{c^2 - v^2\sin^2\theta}\right)\left(v\cos\theta + \sqrt{c^2 - v^2\sin^2\theta}\right)}{\left(-v\cos\theta + \sqrt{c^2 - v^2\sin^2\theta}\right) + \left(v\cos\theta + \sqrt{c^2 - v^2\sin^2\theta}\right)}$$

$$v_{avg} = \frac{\left(c^2 - v^2 \sin^2 \theta - v^2 \cos^2 \theta\right)}{\sqrt{c^2 - v^2 \sin^2 \theta}}$$

$$v_{avg} = \frac{\left(c^2 - v^2\right)}{\sqrt{c^2 - v^2 \sin^2 \theta}}$$

Let $\gamma = v/c$, the above equation becomes

$$v_{avg} = c \cdot \frac{\left(1 - \gamma^2\right)}{\sqrt{1 - \gamma^2 \sin^2 \theta}} \qquad \text{(II)}$$

This formula is our answer to the question.

As

$$\sqrt{1 - \gamma^2 \sin^2 \theta} \geq \sqrt{1 - \gamma^2} \ ,$$

we have

$$v_{avg} \leq c \cdot \frac{\left(1 - \gamma^2\right)}{\sqrt{1 - \gamma^2}}, \text{ or } \ v_{avg} \leq c \cdot \sqrt{\left(1 - \gamma^2\right)} \qquad \text{(III)}$$

From the above, we can get these conclusions:

- For round trips, a swimmer's average swimming speed in flowing water is always smaller than his speed in still water.
- The influence of water is the biggest when the path is parallel to the water's flowing direction, and is the least when the path is perpendicular to it.
- The formula does not work when $v >= c$ (or $\gamma \geq 1$). As a swimmer can never reach upstream in this situation, a meaningful average speed does not exist.